Lecture 10.

A few more words about the norms of operators.

$$\|A\| = \sup_{v \in V} \frac{\|Av\|}{\|v\|} = \sup_{\substack{v \in V \\ \|v\|=1}} \|Av\|.$$

Rmk. Using the last equality we see that the norm is the maximal value of the function

$$f_A : S^{2n-1} \longrightarrow \mathbb{R}, \quad f_A(v) := |Av|, \quad n = \dim V.$$

It is easy to see that f_A is a continuous function. Recall a theorem from calculus that a continuous function on an interval attains its max/min values. Of cource, closed interval! A more general form of this theorem is that if $S \subset \mathbb{R}^m$ is a compact subset and $f: S \to \mathbb{R}$ a continuous function, then f attains max/min values on S. Here "compact" means closed and bounded. That's not the definition, but can be taken as one. Here closed means that every point x of the complement $(\mathbb{R}^n \setminus S)$ satisfies $B_{x,\varepsilon} \subset \mathbb{R}^n \setminus S$ for some $\varepsilon > 0$ ($B_{x,\varepsilon}$ is a ball of radius ε centered at x). It follows that f_A attains a max (as S^{2n-1} is compact), hence, there exists

$$v \in V, \quad |v| = 1, \quad |Av| = \|A\|.$$

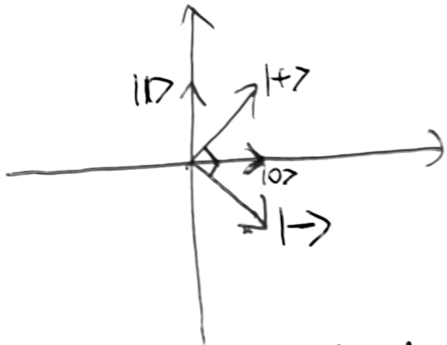
Hadamard operator.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

① $H^2 = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is identity, so $H^{-1} = H$.

$|+\rangle := H(|0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|-\rangle := H(|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$



Let's compute $H^{\otimes n}(|0\ldots0\rangle)$ in the standard basis:

$$H^{\otimes n}|00\ldots0\rangle = |++\ldots+\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \ldots \otimes (|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2^n}} =$$

$$= \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle$$

Rmk. ① We identify $i = i_0 \ldots i_{n-1}$ in the binary form with $|i\rangle$.

② The state $\frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle$ is the superposition of all basic states with equal amplitudes.

What about $H^{\otimes n}(|11\ldots1\rangle)$?

$$H^{\otimes n}(|11\ldots1\rangle) = (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes \ldots \otimes (|0\rangle - |1\rangle) =$$

$$= \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{i_0 + i_1 + \ldots + i_{n-1}} |i\rangle.$$

Finally, let $x = |x_0 \dots x_{n-1}\rangle$, then

$$H^{\otimes n}(|x_0 \dots x_{n-1}\rangle) = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{x \cdot i} |i\rangle,$$

where $x \cdot i = x_0 i_0 + x_1 i_1 + \dots + x_{n-1} i_{n-1}$.